tions into Eqs. (9) and (10) will yield the same frequency equation as Eq. (24).

As pointed out in Ref. 1, the effect of rotatory inertia is small and can be neglected in primarily flexural vibration so that on setting $\bar{\mu} = 0$ in Eq. (24) yields,

$$\Omega^2 = \Omega_0^2 = \theta_1^4[1/(1 + \beta\theta_1^2) + (\delta/2)] \qquad (25)$$

The corresponding result for the straight Bernoulli-Euler beam is $\Omega_0^2 = \theta_1^4$. The first term in Eq. (25) corresponds to Eq. (27) of Ref. 1, where, in the present paper the total span is taken as $2L$. Note that the curvature correction in Eq. (25) is independent of the transverse shear correction and the beam span. A perturbation analysis involving Eq. (24), where $\bar{\mu}$ is regarded as a small parameter gives a first-order correction for rotatory inertia as follows:

$$\Omega^2 = \Omega_0^2[1 - \bar{\mu}\theta_1^2/(1 + \beta\theta_1^2)^2] + 0(\bar{\mu}^2) \qquad (26)$$

where Ω_0^2 is given by Eq. (25). Equation (25) also shows that initial curvature becomes important when $\delta = H^2/r^2 = 0(1)$, or when the rise H, is of the order of the thickness h, of the beam. For a rectangular cross section ($r^2 = h^2/12$) beam, take $E/G = 50$, $h/2L = \frac{1}{10}$, and $H/h = \frac{1}{4}$, so that Eq. (25) yields the result $\Omega^2 = \theta_1^4(\frac{2}{3} + \frac{3}{8}) = \theta_1^4(25/24)$, which demonstrates that even the very slight rise of $H/h = \frac{1}{4}$ is sufficient to cancel the transverse shear effect. In this example $\bar{\mu} = h^2/12L^2 = 0.0033$ which is indeed small compared to unity, so that neglect of rotatory inertia was justified.

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Generalized Finite-Element Method for Compressible Viscous Flow

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IN Ref. 1, Oden derived the generalized finite-element analogue of the Navier-Stokes equations without a variational principle from the energy concept, neglecting thermal effects. Hence he is short of a finite-element analogue for the energy equation in his method. It is the purpose of this Note to derive the generalized finite-element analogue of the Navier-Stokes equations and the energy equation from the residual point of view, while continuity and the equation of state will be omitted for brevity.

Consider the Navier-Stokes equation in Cartesian tensor; i.e.,

$$\rho\left(\frac{\partial w_i}{\partial t} + w_j\frac{\partial w_i}{\partial x_j}\right) - \left(F_i + \frac{\partial p_{ij}}{\partial x_j}\right) = 0 \qquad (1)$$

Following Oden, express every variable in terms of its nodal value of an element dv. For instance,

$$\rho = \rho_M\Psi_M(\mathbf{x}), \quad w_i = w_{Qi}\Psi_Q(\mathbf{x}), \quad R = R_{Si}\Psi_S(\mathbf{x}) \qquad (2)$$

where R is the residual and Ψ_M,Ψ_Q,Ψ_S may be the same "interpolation function" $\Psi(\mathbf{x})$. Summation is implied for every repeated subscript or superscript in each term. M, Q, S, etc., refer to the nodes, while i,j refer to the vector components.

Integrating Eq. (1) with respect to dv of an element, $v^{(e)}$, with weighting function $\Psi_N(\mathbf{x})$, one obtains for the Nth node

$$\rho_M w_{Qi}a_{MQN} + \rho_M w_{Rj}w_{Qi}b_{oojo}{}^{MRQN} - f_{Ni} + \int_{v^{(e)}} p_{ij}\frac{\partial\Psi_N}{\partial x_j}dv = a_{SN}R_{Si} = 0 \qquad (3)$$

where variables are nodal values

$$a_{MQN} = \int_{v^{(e)}}\Psi_M(\mathbf{x})\Psi_Q(\mathbf{x})\Psi_N(\mathbf{x})dv \qquad (4a)$$

$$b_{oojo}{}^{MRQN} = \int_{v^{(e)}}\Psi_M(\mathbf{x})\Psi_R(\mathbf{x})\Psi_{Q,j}(\mathbf{x})\Psi_N(\mathbf{x})d\mathbf{x} \qquad (4b)$$

$$f_{Ni} = \int_{v^{(e)}} F_i\Psi_N(\mathbf{x})dv + \int_{\partial v^{(e)}} p_{ij}n_j\Psi_N(\mathbf{x})dS \qquad (5)$$

$$p_{ik} = -p\delta_{ik} - \frac{2}{3}\mu\frac{\partial w_j}{\partial x_j}\delta_{ik} + \mu\left(\frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i}\right) \qquad (6)$$

For illustration, constant thermal physical constants will be assumed. Thus, one has

$$\int_{v^{(e)}} p_{ij}(\partial\Psi_N/\partial x_j)dv \equiv \int_{v^{(e)}} p_{ik}(\partial\Psi_N/\partial x_k)dv = -p_T\delta_{ik}b_k{}^{NT} - \tfrac{2}{3}\mu w_{Rj}b_{jk}{}^{RN}\delta_{ik} + \mu w_{Qi}b_{kk}{}^{QN} + \mu w_{Qk}b_{ik}{}^{QN} \qquad (7)$$

where

$$b_k{}^{NT} = \int_{v^{(e)}}\Psi_{N,k}\Psi_T dv, \quad b_{jk}{}^{RN} = \int_{v^{(e)}}\Psi_{R,j}\Psi_{N,k}dv \qquad (8a,b)$$

$$b_{ik}{}^{QN} = \int_{v^{(e)}}\Psi_{Q,i}\Psi_{N,k}dv \qquad (8c)$$

It is assumed that a_{SN} is nonsingular, which can be checked in each case, then the nodal values of residual are zero, as are the interpolated residual inside the element. Equation (3) is in agreement with the Navier-Stokes equation analogue derived by Oden, if Ψ_M is taken as unity and $\rho_M = \rho$ for incompressible flow.

Similarly, the energy equation neglecting radiation is (Ref. 2)

$$\frac{\partial Q}{\partial t} + \Phi + \frac{\partial}{\partial x_j}\left(\kappa\frac{\partial T}{\partial x_j}\right) - \rho\left(\frac{\partial e}{\partial t} + w_j\frac{\partial e}{\partial x_j}\right) + p\frac{\partial w_j}{\partial x_j} = 0 \qquad (9)$$

where

$$\Phi = \sigma_{ij}'\partial w_i/\partial x_j, \quad \sigma_{ij}' = 2\mu e_{ij} + \lambda\partial w_j/\partial x_j \qquad (10a,b)$$

$$e_{ij} = \tfrac{1}{2}[(\partial w_i/\partial x_j) + (\partial w_j/\partial x_i)], \quad \lambda = -\tfrac{2}{3}\mu \qquad (10c,d)$$

$$de = C\,dt, \quad C = C_v \qquad (11a,b)$$

Integrating Eq. (9) with the weighting function $\Psi_R(\mathbf{x})$ over the element $v^{(e)}$, one finds

$$a_{HR}\dot{Q}_H + \mu(b_{jjo}{}^{LPR}W_{Li}W_{Pi} + b_{ijo}{}^{MPR}W_{Mj}W_{Pi}) + \lambda b_{jio}{}^{GPR}W_{Gj}W_{Pi} + \int_{\partial v^{(e)}}\Psi_R\kappa(\partial T/\partial x_j)n_j dS - b_{ii}{}^{NR}\kappa T_N - \rho_U C T_N a_{NUR} - \rho_U T_N W_{Pj}b_{ojoo}{}^{UNPR} + p_v W_{Pj}b_{ojo}{}^{VPR} = a_{SR}R_{Si} = 0 \qquad (12)$$

where b's and a's are defined analogous to Eqs. (4a) and (4b), respectively.

Including the analogues of the equation of state and the continuity equation does not complete the formulation for the generalized finite-element method, since there is no variational principle and there are more nodes than elements, in general. But there are more integrated equations than there are unknowns; the equations are applied to the boundary

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nodes, with utilization of the boundary conditions (Ref. 3).

However, the problem is closed if we take the arithmetic average of equations at the nodes. One obtains a valid equation at each node, as the resultant equation remains an integrated differential equation with a weighting function. Then the total number of equations and unknowns are both the number of nodes times the number of equations at each node. This completes the formulation.

In conclusion, it is noted that the advantage of a finite-element method lies in its flexibility in fitting a curved boundary. Whether the present formulations will be of practical value remains to be tested by actual programing of some examples.

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Accuracy of Donnell's Equations for Noncircular Cylinders

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Many papers have considered the stresses and displacements of noncircular cylindrical shells with a variety of edge conditions. Kempner et al.[1–5] have considered the cases of oval cylinders under lateral pressure for simply supported, clamped, and ring-reinforced end conditions where the radius of curvature relation for the oval cross section stated by Marguerre[6] was used as well as equations of the Donnell type. Basuli[7] presented a solution using Donnell type equations with the radius of curvature varying exponentially. It has been shown by Kempner,[8] Hoff,[9] and Morley,[10] however, that the results for circular cylinders when the Donnell equations are used are good only if the cylinder is short. Nevertheless, as Kraus has pointed out,[11] the accuracy of Donnell's assumptions for noncircular cylinders has not been assessed. It is the purpose of the present Note to perform this assessment.

Governing Equations

To investigate the applicability of the Donnell equations when considering noncircular cylindrical shells, the solution found using the equations of the Love-Reissner type will be compared with that obtained using the Donnell equations. The particular case considered for this comparison is an oval cylindrical shell that is simply supported at the edges and has a radius of curvature of the form stated by Marguerre[6]:

$$1/r = (1/r_o)(1 + \xi \cos 4\pi s/L_o) \tag{1}$$

where $r(s)$ is the radius of curvature of the cross section; r_o is the radius of the circle whose perimeter is L_o, the perimeter of the oval cross section; and ξ is the parameter that determines the noncircularity of the oval. To preclude the concave outward case, $0 \leqslant \xi \leqslant 1$. The coordinate system used for such shells is x, s, z where x is in the axial direction, s is in the circumferential direction, and z is the outward normal to the middle surface of the shell, as shown in Fig. 1.

The governing equations for thin cylindrical shells can be reduced to three involving u, v, and w—the translations of the middle surface of the shell in the x, s, and z directions, respectively. These equations are[11]

$$\frac{\partial^2 u}{\partial x^2} - \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u}{\partial s^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial x \partial s} + \frac{\nu}{r}\frac{\partial w}{\partial x} = 0 \tag{2a}$$

$$(\partial^2 v/\partial s^2) + [(1-\nu)/2](\partial^2 v/\partial x^2) + [(1+\nu)/2](\partial^2 u/\partial x \partial s) + (\partial/\partial s)(w/r) + (1/r)(h^2/12)\{(\partial^2/\partial s^2)(v/r) + (1/r) \times [(1-\nu)/2](\partial^2 v/\partial x^2) - (\partial^3 w/\partial s^3) - (\partial^3 w/\partial x^2 \partial s)\} = 0 \tag{2b}$$

$$\nabla^4 w + (1/r)(12/h^2)[\nu(\partial u/\partial x) + (\partial v/\partial s) + (w/r)] - (\partial^3/\partial s^3)(v/r) - (\partial/\partial s)[(1/r)(\partial^2 v/\partial x^2)] = -12q(1-\nu^2)/Eh^3 \tag{2c}$$

when only a uniform lateral external pressure loading q is considered and where ν is the Poisson's ratio of the material, E is its Young's modulus, and h is the shell thickness. ∇^4 is the biharmonic operator.

These equations can be reduced to those attributed to Donnell by assuming that: 1) the transverse shear resultant makes a negligible contribution to the equilibrium of forces in the circumferential direction, and 2) the displacement v negligibly affects the changes in curvature and twist due to the loading. When these two assumptions are made, the three governing equations reduce to[11]

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 u}{\partial s^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial x \partial s} + \frac{\nu}{r}\frac{\partial w}{\partial x} = 0 \tag{3a}$$

$$\frac{\partial^2 v}{\partial s^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2 v}{\partial x^2} + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 u}{\partial x \partial s} + \frac{\partial}{\partial s}\left(\frac{w}{r}\right) = 0 \tag{3b}$$

$$\nabla^4 w - \frac{12}{rh^2}\left(\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial s} + \frac{w}{r}\right) = -\frac{12q(1-\nu^2)}{Eh^3} \tag{3c}$$

Solution

To compare the two sets of equations—those of the Love-Reissner type with those of the Donnell type—a simply supported oval cylindrical shell is analyzed in detail. The boundary conditions at $x = 0$ and $x = L$ for such a situation are

$$w = v = \partial u/\partial x = \partial^2 w/\partial x^2 = 0 \tag{4}$$

With these boundary conditions, the solutions for the displacements can be assumed to be of the form

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=0,4,\ldots}^{\infty} \left(\frac{qr_o^2}{Eh}\right) \begin{Bmatrix} A_{mn} \cos\dfrac{m\pi x}{L} \cos\dfrac{n\pi s}{L_o} \\ B_{mn} \sin\dfrac{m\pi x}{L} \sin\dfrac{n\pi s}{L_o} \\ C_{mn} \sin\dfrac{m\pi x}{L} \cos\dfrac{n\pi s}{L_o} \end{Bmatrix} \tag{5}$$

when the loading is expanded in the form:

$$\frac{12q(1-\nu^2)}{Eh^3} = \sum_{m=1,3,\ldots}^{\infty} \left(\frac{qr_o^2}{Eh}\right)\left[\frac{48(1-\nu^2)}{mh^2 r_o^2}\right] \sin\frac{m\pi x}{L} \tag{6}$$

Because of the noncircularity of the cross section, it is not possible to assume an exponential form of solution and compare the roots of the complementary equation as was done for the circular case.[8–11] Therefore, the method of solution is to substitute the assumed solutions in Eqs. (2) and (3) and, using Eq. (1) for the radius of curvature relation, solve

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